Framework for Mathematical Proficiency for Teaching

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Secondary school mathematics comprises far more than facts, routines, and strategies. It includes a vast array of interrelated mathematical concepts, ways to represent and communicate those concepts, and tools for solving all kinds of mathematical problems. It requires reasoning and creativity, providing learners with mathematical knowledge while also laying a foundation for further studies in mathematics and other disciplines.

To facilitate the learning of secondary school mathematics, teachers need a particular kind of proficiency. Mathematical proficiency for teaching at the secondary level is the mathematical expertise and skill a teacher has and uses for the purpose of promoting students' understanding of, proficiency with, and appreciation for mathematics. It requires that teachers not only know more mathematics than they teach but also know it more deeply.

Mathematical proficiency for teaching (MPT) is unique to the work of teaching. It is different from the mathematical proficiency needed for engineering, accounting, or the medical professions. It is even different from the mathematical proficiency a mathematician needs. For example, a mathematician may prove a theorem, and an architect may perform geometric calculations. For these users of mathematics, it is sufficient that they have the skills and understanding for the task at hand. But a teacher's work includes these tasks as well as interpreting students' mathematics, developing multiple representations of a mathematical concept, knowing where students are on the path of mathematical understanding, and so on.

Mathematical proficiency for teaching is *dynamic*. We make a distinction between knowledge and proficiency. Knowledge may be seen as static and something that cannot be directly observed, whereas proficiency can be viewed as the dynamic use of the knowledge one has. Proficiency can be observed in a teacher's actions and the decisions he or she makes. Also, because of its dynamic nature, MPT grows and deepens in the course of a teacher's career.

The focus of our MPT framework is on *secondary school* mathematics. That is, we seek to characterize the mathematical proficiency that is useful to secondary teachers as distinct from the proficiency needed by elementary school mathematics teachers. We believe that MPT for secondary school is different from MPT for elementary school in at least four ways: (1) There is a wider range of mathematics content (i.e., more topics are studied); (2) there is a greater emphasis on formality, axiomatic systems, and rigor in regard to mathematical proof; (3) there is more explicit attention to mathematical structure and abstraction (e.g., identities, inverses, domain, and undefined elements); and (4) the cognitive development of secondary students is such that they can reason differently from elementary school children about such matters as proportionality, probability, and mathematical induction.

Our framework has been developed out of *classroom practice,* and we have drawn examples from a wide variety of classroom contexts. We have examined episodes occurring in the work of prospective and practicing secondary mathematics teachers and mathematics educators at the college level. From this collection, we have determined elements of mathematics proficiency that would be beneficial to secondary mathematics teachers. We describe a wide sample, as opposed to a comprehensive catalog, of mathematical proficiency for teaching that comes from our analyses of these classroom episodes.

Mathematical proficiency for teaching is not the same as proficiency in pedagogy. Being equipped with the proficiency described in our MPT framework is not simply a matter of "knowing the mathematics" plus "knowing how to teach." The task of teaching mathematics cannot be partitioned into such simple categories.

A Framework for MPT

Mathematical proficiency for teaching (MPT) can be viewed as having three overlapping components: mathematical proficiency, mathematical activity, and mathematical work of teaching (Figure 1). Each component emphasizes a different aspect of MPT. MPT is a developing quality and not an endpoint.

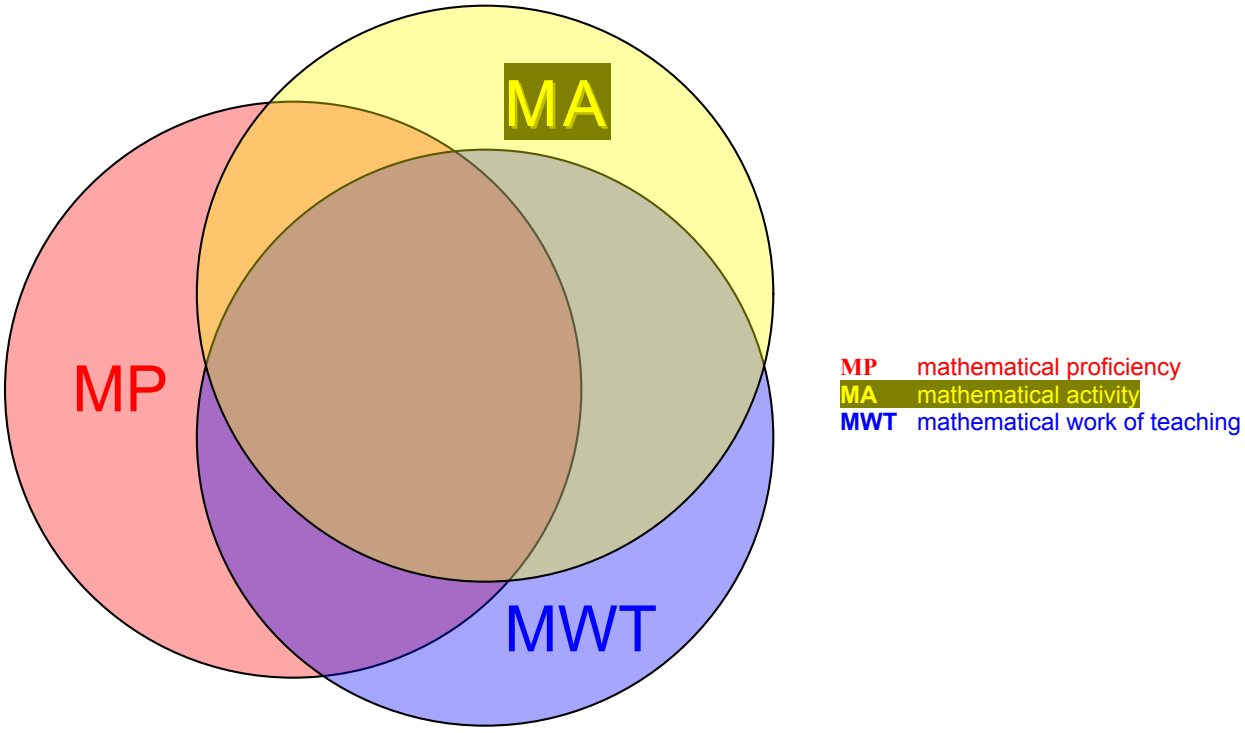

Figure 1. Three components of mathematical proficiency for teaching.

Mathematical proficiency includes aspects of mathematical knowledge and ability, such as conceptual understanding and procedural fluency, that teachers need themselves and that they seek to foster in their students. The mathematical proficiency

teachers need goes well beyond what one might find in secondary students or even the average educated adult. Students' development of mathematical proficiency usually depends heavily on how well developed their teacher's proficiency is. Secondary teachers of mathematics need proficiency with the mathematics their students should have learned in elementary school, and they need proficiency with the mathematics their students may encounter when taking mathematics and related subjects in college.

Proficiency in mathematical activity can be thought of as "doing mathematics." It is like, and overlaps, the mathematical proficiency component, but the emphasis is on those mathematical activities that teachers employ and that they want their students to learn. Other users of mathematics may engage in such activities from time to time, but teachers need a more conscious, elaborated command of their nature and particulars. Examples include representing mathematical objects and operations, connecting mathematical concepts, modeling mathematical phenomena, and justifying mathematical arguments. This facet of mathematical proficiency for teaching is on display as teachers engage students in the day-to-day study of mathematics. Teachers need deep knowledge, for example, of what characterizes the structure of mathematics (as opposed to conventions that have been adopted over the centuries) and how to generalize mathematical findings. The more a teacher's proficiency in mathematical activity has developed, the better equipped he or she will be to facilitate the learning and doing of mathematics.

Proficiency in the mathematical work of teaching diverges sharply from the mathematical proficiency needed in other professions requiring mathematics. One of its aspects is an understanding of the mathematical thinking of students, which may include, for example, recognizing the mathematical nature of their errors and misconceptions. Teachers need to be able to decide whether a proof might be circular or incomplete, how well a proposed solution satisfies the conditions of a problem, and whether an alternative definition is equivalent to one already proposed. Another aspect of the mathematical work of teaching is knowledge of and proficiency in the mathematics that comes before and after what is being studied currently. A teacher benefits from knowing what students have learned in previous years so that he or she can help them build upon that prior knowledge. The teacher also needs to provide a foundation for the mathematics they will be learning later, which requires knowing and understanding the mathematics in the rest of the curriculum.

The three components of MPT—mathematical proficiency, mathematical activity, and mathematical work of teaching—together form a full picture of the mathematics required of a teacher of secondary mathematics. It is not enough to know the mathematics that students are learning. Teachers must also possess a depth and extent of mathematical proficiency that will equip them to foster their students' mathematical proficiency. Mathematical proficiency informs the other two perspectives on MPT: Mathematical activity and the mathematical work of teaching emerge from, and depend upon, the teacher's mathematical proficiency.